Pion-Nucleon Vertex Functions*

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A quantitative study of the pion-nucleon vertex function with one nucleon off the mass shell is presented using dispersion-theoretical techniques. The discussion is based on the unitarity conditions for the vertex function and for the one-nucleon irreducible parts of the pion-nucleon scattering amplitudes with $I=J=1/2$. It is shown in terms of the N/D method that the vertex function has a sharp maximum in the low-energy region. Its existence is found to be crucial for a certain inequality, which is due to the requirement of no ghosts in the theory, to be satisfied. The nucleon propagator is strongly suppressed at the energy of this maximum. Reasonable results obtained seem to suggest that ghosts are not present in pion physics, although the present calculation is still incomplete. The pion-nucleon vertex function with pion off the mass shell is also discussed and some early attempts are re-examined. The importance of a pole or a resonance-like behavior in this vertex is suggested.

I. INTRODUCTION AND SUMMARY

APART from perturbation expansion, which is certainly wrong for strong interactions, there does not seem to exist any widely accepted method of calculating the propagators of physical particles and their proper vertex functions. Although some nonperturbational attempts[1,2] have been made on the basic Green functions, the ladder approximations adopted in these works are, even qualitatively, very inadequate. As will be seen later, the pion-nucleon vertex function of Federbush *et al.*,[2] for example, badly violates the inequality of Lehmann, Symanzik, and Zimmermann[3] (LSZ), which is essentially due to the requirement of no ghost states in a theory.

The aim of the present work is to present a new method of studying the nucleon propagator and the pion-nucleon vertex function with one nucleon off the mass shell, in which we make repeated use of S-matrix theoretical techniques. Crucial to our approach are the two kinds of unitarity relations investigated in a previous work.[4]

The P-wave pion-nucleon scattering amplitude with $I=J=\frac{1}{2}$ is divided into two parts, the contribution from the one-nucleon intermediate state with all the radiative corrections included and the rest which we call the one-nucleon irreducible term. The fact that the second term satisfies unitarity by itself[4] enables us to calculate it in terms of the N/D method.[5] It is shown that the one-nucleon irreducible term has a resonance-like behavior in the low-energy region. In order to avoid confusion with ordinary resonances, our resonance will be called a *pseudoresonance* in this paper. By unitarity, the vertex function must have a resonance behavior at the same energy. The existence of this pseudoresonance in the low-energy region turns out to be vital for the LSZ inequality to be satisfied. Although we had to make an arbitrary choice for the high-energy behavior of the phases of the vertex function, it should be mentioned that the contribution from the pion-nucleon intermediate states to the LSZ sum rule comes dominantly from the low-energy region where we know more information is available.

In Sec. II we discuss general properties of the nucleon propagator and the pion-nucleon vertex function with one nucleon off the mass shell. In order to reduce the complication due to the spin of the nucleon, we shall take advantage of the Gell-Mann–Low form[6] of the representation for the nucleon propagator. The results obtained in I for the pion case are extended to the nucleon case.

In Sec. III we calculate the one-nucleon irreducible part of the P-wave scattering amplitude with $I=J=\frac{1}{2}$ in terms of the N/D method, in which the forces due to the nucleon exchange and the 3-3 isobar exchange are considered. We get a pseudoresonance at the energy $w\approx m+2\mu$, where w denotes the invariant total energy. The phase $\eta_P(w)$ of this one-nucleon irreducible term is, by unitarity, equal to the phase $\eta_+(w)$ of the vertex function in the energy region $m+\mu\leq w\leq m+2\mu$. The vertex function is then calculated in terms of an Omnès integral,[7] by replacing $\eta_+(w)$ with $\eta_P(w)$ in the low-energy region and by assuming an arbitrary, but not unreasonable, behavior for $\eta_+(w)$ in the high-energy region and for $\eta_-(w)$, the phase of the vertex function on its left-hand cut. Numerical calculation shows that the pion-nucleon contribution to the LSZ sum rule is about 0.8, 0.73 from the right-hand cut and 0.07 from the left-hand cut, which leaves 0.2 to Z_N plus the contributions from the inelastic states. The nucleon propagator in the low-energy region is then calculated from this vertex function. It is found that the propagator is strongly suppressed at the pseudoresonance energy. Although our calculation is still very crude, reasonable

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[1] S. F. Edwards, Phys. Rev. **90**, 284 (1953).
[2] P. Federbush, M. L. Goldberger, and S. B. Treiman, Phys. Rev. **112**, 642 (1958).
[3] H. Lehmann, K. Symanzik, and W. Zimmermann, Nuovo Cimento **2**, 425 (1955).
[4] M. Ida, Phys. Rev. **135**, B499 (1964). Hereafter this paper will be referred to as I.
[5] G. F. Chew and S. Mandelstam, Phys. Rev. **119**, 467 (1960).
[6] M. Gell-Mann and F. E. Low, Phys. Rev. **95**, 1300 (1954).
[7] R. Omnès, Nuovo Cimento **8**, 316 (1958).

results obtained may suggest that pion physics does not have a ghost state.

A few remarks are given in Sec. IV on some approaches to the pion-nucleon vertex function $\Gamma(s)$ with pion off the mass shell. In terms of the method of Geshkenbein and Ioffe[8] and Meyman[9] it is shown that we obtain an upper bound of about 2.0 on the pion-nucleon coupling constant $g^2/4\pi(\approx 15)$ if we neglect the cut of $\Gamma(s)$ between $9\mu^2$ and $4m^2$ and if $\Gamma(s)$ has no poles in the s plane with the cut extending from $4m^2$ to $+\infty$. It follows that the $\Gamma(s)$ of Federbush *et al.*, for instance, makes the pion-nucleon contribution to the LSZ sum rule larger than 7.5 when $g^2/4\pi$ is given experimentally. This awful violation of the LSZ inequality necessarily leads to a ghost state. The prescription of Redmond[10] to eliminate ghosts is then discussed. It is found that his procedure introduces a pole in $\Gamma(s)$, which strongly damps the original vertex in the physical region. However, the form factor $K(s)$ remains unchanged and hence retains a ghost pole. Putting these considerations together, we conclude that the LSZ inequality requires a strong suppression of the vertex function, which may probably be caused by a pole or a pseudoresonance in $\Gamma(s)$ lying below $4m^2$.

In the Appendix we rederive the upper bound on the pion-nucleon coupling constant obtained by Geshkenbein and Ioffe (GI)[8] from the LSZ inequality and some additional assumptions. The kinematics we have adoped in Sec. II permit a considerable simplification of their original derivation and of their final expression. The reason for this is that we consider a single function of w with one subsidiary condition, while they did two functions of w^2 with three conditions.

II. GENERAL PROPERTIES OF THE NUCLEON PROPAGATOR AND OF THE PION-NUCLEON VERTEX FUNCTION

Much of what we discuss in this section is perhaps well known. However, we shall put an emphasis on a kind of reflection symmetry characteristic of the fermion case. For this purpose we make use of the Gell-Mann–Low representation for the nucleon propagator,[6]

$$iS'_F(-ip)=\frac{1}{m+ip}+\frac{1}{\pi}\int_{m+\mu}^{\infty}\frac{\sigma_+(w')}{w'+ip}dw' -\frac{1}{\pi}\int_{m+\mu}^{\infty}\frac{\sigma_-(w')}{w'-ip}dw', \quad (2.1)$$

which has a definite advantage in its physical clearness over others. Since

$$\sigma_\pm(w)\geq 0 \quad \text{for} \quad w\geq m+\mu,$$

the function

$$iS'_F(w)=\frac{1}{m-w}+\frac{1}{\pi}\int_{m+\mu}^{\infty}\frac{\sigma_+(w')}{w'-w}dw' -\frac{1}{\pi}\int_{m+\mu}^{\infty}\frac{\sigma_-(w')}{w'+w}dw', \quad (2.2)$$

obtained by replacing $-ip$ in Eq. (2.1) with w is a Herglotz function.[11] We introduce a renormalization function by

$$Z_N^{-1}(w)\equiv S_F'(w)/S_F(w), \quad (2.3)$$

which is, by definition, normalized to unity at $w=m$.

It is easy to see that

$$S_F'(-ip)\cdot S_F^{-1}(-ip) =\Lambda_+(p)Z_N^{-1}(w)+\Lambda_-(p)Z_N^{-1}(-w), \quad (2.4)$$

where the $\Lambda_\pm(p)$ are the projection operators to positive and negative states, respectively:

$$\Lambda_\pm(p)=(w\mp ip)/2w, \quad (2.5)$$

and $w=(-p^2)^{1/2}$, with $p^2\equiv p_1^2+p_2^2+p_3^2-p_0^2$. Because of the Herglotz property of the propagator, $Z_N(w)$ can be written as

$$Z_N(w)=1+\frac{w-m}{\pi}\int_{m+\mu}^{\infty}\frac{\tau_+(w')}{w'-w}dw' -\frac{w-m}{\pi}\int_{m+\mu}^{\infty}\frac{\tau_-(w')}{w'+w}dw'+(w-m)\sum_n\frac{C_n}{w_n-w}, \quad (2.6)$$

where

$$\tau_\pm(w)\equiv|Z_N(\pm w)|^2\sigma_\pm(w) \quad (2.7)$$

and $C_n\geq 0$. The sum on the right-hand side of Eq. (2.6) represents Castillejo-Dalitz-Dyson (CDD) terms.[11] By taking the limit, $|w|\to\infty$, we obtain the LSZ sum rule

$$1=\frac{1}{\pi}\int_{m+\mu}^{\infty}[\tau_+(w)+\tau_-(w)]dw+Z_N+\sum_n C_n, \quad (2.8)$$

where Z_N,

$$Z_N=\lim_{|w|\to\infty}Z_N(w), \quad (2.9)$$

is the nucleon wave-function renormalization constant and satisfies the inequalities

$$1>Z_N\geq 0.$$

By dropping off the non-negative terms from Eq. (2.8), we obtain

$$1\geq\frac{1}{\pi}\int_{m+\mu}^{\infty}[\tau_+(w)+\tau_-(w)]dw, \quad (2.10)$$

[8] B. V. Geshkenbein and B. L. Ioffe, Zh. Eksperim. i Teor. Fiz. **44**, 1211 (1963); **45**, 555 (1963) [English transls.: Soviet Phys.—JETP **17**, 820 (1963); **18**, 382 (1964)].

[9] N. N. Meyman, Zh. Eksperim. i Teor. Fiz. **44**, 1228 (1963) [English transl.: Soviet Phys.—JETP **17**, 830 (1963)].

[10] P. J. Redmond, Phys. Rev. **112**, 1404 (1958).

[11] L. Castillejo, R. H. Dalitz, and F. J. Dyson, Phys. Rev. **101**, 453 (1956).

from which it follows that

$$\lim_{w\to\infty} w\tau_\pm(w)=0. \qquad (2.11)$$

For completeness, we list here a formal expression for the bare mass m_0 of the nucleon[12] in our notation:

$$m_0=\left\{m+\int_{m+\mu}^{\infty} w[\sigma_+(w)-\sigma_-(w)]dw\right\}$$

$$\times\left\{1+\int_{m+\mu}^{\infty}[\sigma_+(w)+\sigma_-(w)]dw\right\}^{-1}$$

$$=Z_N\left\{m+\int_{m+\mu}^{\infty} w[\sigma_+(w)-\sigma_-(w)]dw\right\}. \qquad (2.12)$$

It is easy to see that the bare mass is the average of w's with the weight function,

$$\delta(w-m)+\theta(w-m-\mu)\sigma_+(w)+\theta(-w-m-\mu)\sigma_-(|w|).$$

In the lowest order perturbation theory the two spectral functions asymptotically become the same for large w:

$$\sigma_+(w)\sim\sigma_-(w).$$

This asymptotic equality reduces the divergence of the self-mass of the nucleon from a linear to a logarithmic one.

We express the form factor or the improper vertex function with one nucleon off the mass shell in the form

$$\bar{u}(p')\tau_\alpha i\gamma_5 K(-ip)=\bar{u}(p')\tau_\alpha i\gamma_5[\Lambda_+(p)K(w)+\Lambda_-(p)K(-w)]. \qquad (2.13)$$

$K(w)$ is normalized to g at $w=m$, where $g^2/4\pi=15$. The contributions to the spectral functions $\sigma_\pm(w)$ from the pion-nucleon intermediate states are then given by

$$\sigma_\pm{}^{N\pi}(w)=\frac{3}{16\pi}\frac{\rho_\pm(w)}{(w\mp m)^2}|K(\pm w)|^2, \qquad (2.14)$$

where

$$\begin{aligned}\rho_\pm(w)&=q(w)\{(w\mp m)^2-\mu^2\}/w^2\\ &=\{(w\mp m)^2-\mu^2\}^{3/2}\cdot\{(w\pm m)^2-\mu^2\}^{1/2}/2w^3\\ &=\{1-[(m+\mu)/w]^2\}^{1/2}\{1-[(m-\mu)/w]^2\}^{1/2}\\ &\quad\times\{(w\mp m)^2-\mu^2\}/2w, \qquad (2.15)\end{aligned}$$

and $q(w)$ is the center-of-mass momentum:

$$q(w)=\{(w-m)^2-\mu^2\}^{1/2}\{(w+m)^2-\mu^2\}^{1/2}/2w. \qquad (2.16)$$

The numerical factor of 3 in Eq. (2.14) is due to isotopic spin.

Now we define the proper vertex function by

$$\Gamma(w)\equiv Z_N(w)K(w). \qquad (2.17)$$

The contributions to $\tau_\pm(w)$ from the pion-nucleon intermediate states are found to be

$$\tau_\pm{}^{N\pi}(w)=\frac{3}{16\pi}\frac{\rho_\pm(w)}{(w\mp m)^2}|\Gamma(\pm w)|^2. \qquad (2.18)$$

Because of the LSZ inequality

$$1\geq\frac{1}{\pi}\int_{m+\mu}^{\infty}[\tau_+{}^{N\pi}(w)+\tau_-{}^{N\pi}(w)]dw, \qquad (2.19)$$

which can be obtained from the inequality (2.10), we have

$$\lim_{w\to\infty} w\tau_\pm{}^{N\pi}(w)=0, \qquad (2.20)$$

and hence[13]

$$\lim_{w\to\infty}\Gamma(\pm w)=0. \qquad (2.21)$$

Next we discuss the unitary requirement on the proper and improper vertex functions. The partial-wave amplitudes in the pion-nucleon scattering states with $I=J=\frac{1}{2}$ will be denoted by $f_P(w)$ and $f_S(w)$ for the P and S waves, respectively. By unitarity, they can be expressed in the form

$$f_P(w)=(e^{i\delta}P\sin\delta_P)/q, \qquad (2.22a)$$

$$f_S(w)=(e^{i\delta}S\sin\delta_S)/q, \qquad (2.22b)$$

where the phase shifts are real up to the first inelastic threshold, $m+2\mu$. The unitarity also gives us the relations[14]

$$\mathrm{Im}K(w_+)=q(w_+)f_P{}^*(w_+)K(w_+), \qquad (2.23a)$$

$$\mathrm{Im}K(-w_+)=q(-w_+)f_P{}^*(-w_+)K(-w_+), \qquad (2.23b)$$

for w, $m+\mu\leq w\leq m+2\mu$, where $w_+=w+i\epsilon$ and ϵ is an infinitesimally small positive number. By making use of the MacDowell relation[15]

$$f_P(-w)=-f_S(w), \qquad (2.24)$$

we can rewrite Eq. (2.23b) as

$$\mathrm{Im}K(-w_+)=q(w_+)f_S{}^*(w_+)K(-w_+). \qquad (2.23b')$$

Here we have taken the cuts of $q(w)$ in the w plane between $-(m+\mu)$ and $-(m-\mu)$ and between $m-\mu$ and $m+\mu$, so that $q(w)$ is real for w, $|w|>m+\mu$, and $q(-w)=-q(w)$ when $w>m+\mu$.

By assuming ordinary analyticity for $K(w)$, it can be expressed, apart from possible zeros, in the form

$$K(w)=g\exp\left[\frac{w-m}{\pi}\int_{m+\mu}^{\infty}\frac{\delta_+(w')dw'}{(w'-m)(w'-w)}-\frac{w-m}{\pi}\int_{m+\mu}^{\infty}\frac{\delta_-(w')dw'}{(w'+m)(w'+w)}\right], \qquad (2.25)$$

[12] H. Lehmann, Nuovo Cimento 11, 342 (1954).

[13] E. Ferrari and G. Jona-Lasinio, Nuovo Cimento 10, 310 (1958).

[14] S. Okubo, R. E. Marshak, and E. C. G. Sudarshan, Phys. Rev. 113, 944 (1959); I. Umemura and K. Watanabe, Progr. Theoret. Phys. (Kyoto) 29, 893 (1963).

[15] S. W. MacDowell, Phys. Rev. 116, 774 (1959).

where $\delta_+(w)$ and $\delta_-(w)$ are the real phases of $K(w)$ and $K(-w)$ for $w \geq m+\mu$, respectively, and we have

$$\delta_+(w)=\delta_P(w)\,, \tag{2.26a}$$

$$\delta_-(w)=\delta_S(w)\,, \tag{2.26b}$$

for w, $m+\mu \leq w \leq m+2\mu$.

Following the previous work, we divide the partial-wave amplitudes into the two parts, the one-nucleon reducible term and the one-nucleon irreducible term:

$$f_P(w)=\frac{3}{16\pi}\frac{(w-m)^2-\mu^2}{w^2}\times\Gamma(w)\frac{Z_N^{-1}(w)}{m-w}\Gamma(w)+g_P(w)\,, \tag{2.27a}$$

$$f_S(w)=-\frac{3}{16\pi}\frac{(w+m)^2-\mu^2}{w^2}\times\Gamma(-w)\frac{Z_N^{-1}(-w)}{m+w}\Gamma(-w)+g_S(w)\,. \tag{2.27b}$$

The unitarity for the vertex function can be expressed in terms of $g_P(w)$ and $g_S(w)$ as

$$\mathrm{Im}\Gamma(w_+)=q(w_+)g_P{}^*(w_+)\Gamma(w_+)\,, \tag{2.28a}$$

$$\mathrm{Im}\Gamma(-w_+)=q(w_+)g_S{}^*(w_+)\Gamma(-w_+)\,, \tag{2.28b}$$

for w, $m+\mu \leq w \leq m+2\mu$, which correspond to Eqs. (2.23a) and (2.23b′), respectively. We can also show that $g_P(w)$ and $g_S(w)$ satisfy the unitarity by themselves. Therefore, we can write them as

$$g_P(w)=(e^{i\eta_P}\sin\eta_P)/q\,, \tag{2.29a}$$

$$g_S(w)=(e^{i\eta_S}\sin\eta_S)/q\,, \tag{2.29b}$$

where the phases, $\eta_P(w)$ and $\eta_S(w)$, are real up to the first inelastic threshold. $\Gamma(w)$ can be written in a form corresponding to Eq. (2.25),

$$\Gamma(w)=g\exp\left[\frac{w-m}{\pi}\int_{m+\mu}^{\infty}\frac{\eta_+(w')dw'}{(w'-m)(w'-w)}-\frac{w-m}{\pi}\int_{m+\mu}^{\infty}\frac{\eta_-(w')dw'}{(w'+m)(w'+w)}\right], \tag{2.30}$$

apart from possible zeros and poles. Here we have

$$\eta_+(w)=\eta_P(w)\,, \tag{2.31a}$$

$$\eta_-(w)=\eta_S(w)\,, \tag{2.31b}$$

for w, $m+\mu \leq w \leq m+2\mu$.

Although the phases of the f's and the g's become complex for w above $m+2\mu$, their imaginary parts must be positive, because it follows from the unitarity condition that

$$q|f_P(w)|\leq 1\,,\quad q|g_P(w)|\leq 1\,, \tag{2.32a}$$

$$q|f_S(w)|\leq 1\,,\quad q|g_S(w)|\leq 1\,, \tag{2.32b}$$

for w, $w\geq m+\mu$. We find from these inequalities and Eqs. (2.27a) and (2.27b) that

$$|Z_N(\pm w)|/|Y(\pm w)|^2 \geq (3g^2/32\pi)\rho_\pm(w)/(w\mp m)\,, \tag{2.33}$$

for w, $w>m+\mu$, where $Y(w)$ is defined by

$$Y(w)\equiv\Gamma(w)/g\,. \tag{2.34}$$

Taking the limit, $w\to\infty$, we obtain

$$\lim_{w\to\infty}\frac{|Z_N(\pm w)|}{|Y(\pm w)|^2}\geq(3/16)g^2/4\pi\,. \tag{2.35}$$

III. NUMERICAL CALCULATION OF THE VERTEX FUNCTION AND THE PROPAGATOR

This section is devoted to a numerical analysis of the pion-nucleon vertex function with one nucleon off the mass shell as well as the nucleon propagator. We have seen in the previous section that in the low-energy region $\eta_+(w)$ is essentially equal to the phase $\eta_P(w)$ of $g_P(w)$. Since $g_P(w)$ satisfies unitarity by itself, we will calculate it by the N/D method.

Following Frautschi and Walecka,[16] we introduce a function defined by

$$h_P(w)\equiv(e^{i\eta_P}\sin\eta_P)/\rho_P(w)\,, \tag{3.1}$$

where

$$\rho_P(w)\equiv(m/\mu)^2\rho_+(w)\,. \tag{3.2}$$

The factor $(m/\mu)^2$ is merely for easier correspondence with the nonrelativistic limit. The forces we consider here are due to the nucleon exchange and the 3-3 resonance exchange. For simplicity, we shall adopt pole approximation for these forces. We express $h_P(w)$ in the form

$$h_P(w)=N(w)/D(w) \tag{3.3}$$

with

$$D(w)=1-\frac{w-m}{\pi}\int_{m+\mu}^{\infty}\frac{\rho_P(w')N(w')}{(w'-m)(w'-w)}dw'\,. \tag{3.4}$$

We approximate $N(w)$ by

$$N(w)=\frac{1}{3}\frac{f^2}{w-m}+\frac{16}{9}\frac{f^{*2}}{w-w_1}D(w_1)\,, \tag{3.5}$$

where

$$f^2\equiv(\mu/2m)^2g^2/4\pi=0.08\,.$$

[16] S. C. Frautschi and J. D. Walecka, Phys. Rev. **120**, 1486 (1960).

For numerical calculation, we shall take the value $0.68m$ for w_1 and use the relation[17]

$$f^{*2}=\tfrac{3}{2}f^2. \qquad (3.6)$$

The Eqs. (3.4) and (3.5) can easily be solved and the solution is given by

$$D(w)=1-\frac{f^2}{3}\frac{w-m}{\pi}\int_{m+\mu}^{\infty}\frac{\rho_P(w')dw'}{(w'-m)^2(w'-w)}-D(w_1)\frac{16f^{*2}}{9}\frac{w-m}{\pi}\times\int_{m+\mu}^{\infty}\frac{\rho_P(w')dw'}{(w'-m)(w'-w_1)(w'-w)}, \qquad (3.7)$$

where

$$D(w_1)=\left[1+\frac{f^2}{3}\frac{m-w_1}{\pi}\int_{m+\mu}^{\infty}\frac{\rho_P(w)dw}{(w-m)^2(w-w_1)}\right]\times\left[1-\frac{16f^{*2}}{9}\frac{m-w_1}{\pi}\times\int_{m+\mu}^{\infty}\frac{\rho_P(w)dw}{(w-m)(w-w_1)^2}\right]^{-1}\approx 1.56. \qquad (3.8)$$

The phase shift $\eta_P(w)$ can be obtained from the expression

$$\rho_P(w)N(w)\cot\eta_P(w)=\mathrm{Re}D(w). \qquad (3.9)$$

The solid part of the curve for $\eta_+(w)$ in Fig. 1 shows our numerical result for $\eta_P(w)$. It has a pseudoresonance at w, $w\approx m+2.1\mu$. This resonance is different from ordinary ones in that it cannot dirrectly be observed by scattering experiments. However, it plays a very important role in determining the behavior of the vertex function, as will be seen shortly.

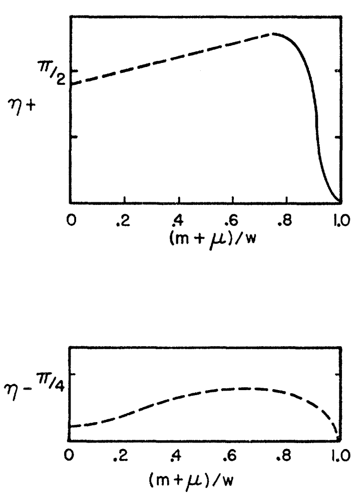

FIG. 1. The phases $\eta_\pm(w)$ used to calculate $Y(w)$. The solid part of the curve for $\eta_+(w)$ is equal to the $\eta_P(w)$ calculated in terms of the N/D method.

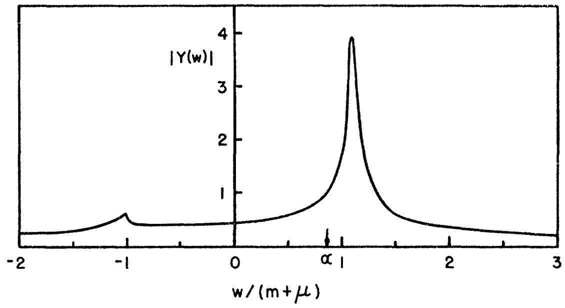

FIG. 2. The behavior of $|Y(w)|$. Its maximum is at the energy $w\approx m+1.8\,\mu$. α stands for $m/(m+\mu)$.

As for the $\eta_-(w)$, it is essentially given in the low-energy region by $\eta_S(w)$, which is to be calculated by the N/D method. Unfortunately, a reliable calcualtion of $\eta_S(w)$ is much more difficult to perform than that of $\eta_P(w)$ is. In the $I=J=\frac{1}{2}$ S-wave state, the nucleon-exchange and the 3-3 resonance-exchange terms together give rise to a strongly attractive force. There is another term characteristic of our approach, which is due to the pseudoresonance. It gives a strongly repulsive force and largely diminishes the attractive force mentioned above. There should be more forces, the pion-pion force for example, which are important in the determination of $\eta_S(w)$. It is beyond the scope of the present work to investigate them quantitatively. We shall be satisfied in this paper by making an arbitrary choice for $\eta_-(w)$ in the low-energy region. It will be seen later, however, that the contribution to the LSZ sum rule from this region is very small. Although a dynamical calculation of $\eta_-(w)$ is certainly desirable, we hope that our qualitative result will remain unchanged.

We shall make the $\eta_P(w)$ calculated above replace $\eta_+(w)$ in the energy range up to $m+2.5\mu$, where the former shows flattening. We lack knowledge of the high-energy behaviors of $\eta_+(w)$ and $\eta_-(w)$. However, they should be such that they make the high-energy contribution to the LSZ sum rule small. Otherwise the LSZ inequality would be violated because, as it turns out, the low-energy contribution is large by itself. It will be assumed that the phases of $Y(w)$ become asymptotically the same in the high-energy region as those of $Y_{\mathrm{GI}}(w)$, which minimize the integral

$$\Phi(Y)=\frac{3}{4\pi}\left\{\int_{m+\mu}^{\infty}\frac{\rho_+(w)}{(w-m)^2}|Y(w)|^2dw+\int_{m+\mu}^{\infty}\frac{\rho_-(w)}{(w+m)^2}|Y(-w)|^2dw\right\}. \qquad (3.10)$$

A more detailed discussion of $Y_{\mathrm{GI}}(w)$ is given in the Appendix, which deals with finding an upper bound on the coupling constant. We mention here only that $(g^2/4\pi)\Phi$ gives the pion-nucleon contribution to the

[17] D. Amati and S. Fubini, Ann. Rev. Nucl. Sci. 12, 359 (1962).

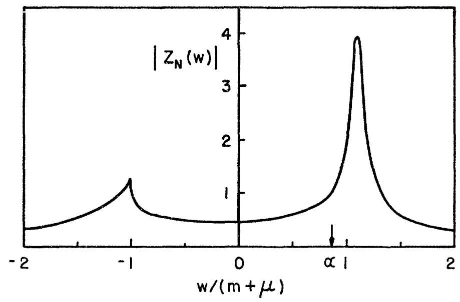

FIG. 3. The behavior of $|Z_N(w)|$. α stands for $m/(m+\mu)$.

LSZ sum rule and hence must be smaller than unity (the LSZ inequality). Our $\eta_+(w)$ and $\eta_-(w)$, which will be used to calculate $\Gamma(w)$, are shown in Fig. 1.

In calculating the vertex function from Eq. (2.30), we shall assume that it has neither zeros nor poles in the w plane with the cuts. The validity of the assumption is to be judged from the reasonableness of a result which follows. The $\Gamma(w)$ thus obtained is shown in Fig. 2. It has a sharp peak in the low-energy region of the right-hand cut, which is due to the pseudoresonance. Our primary concern is whether our $\Gamma(w)$ satisfies the LSZ inequality or not. We find

$$\frac{1}{\pi}\int_{m+\mu}^{\infty}\tau_+^{N\pi}(w)dw+\frac{1}{\pi}\int_{m+\mu}^{\infty}\tau_-^{N\pi}(w)dw =0.73+0.07 =0.80. \qquad (3.11)$$

We see that the LSZ inequality is satisfied. Our result leaves 0.20 to Z_N plus the inelastic contributions in the LSZ sum rule. It is to be mentioned here that most contribution in (3.11) comes from the $\tau_+(w)$ in the low energy region, where our $\Gamma(w)$ is more reliable than elsewhere.

What role does the pseudoresonance play in our calculation? If we do not accept the existence of the pseudoresonance in the low-energy region, then the original sharp peak of $\Gamma(w)$ becomes a broad bump and the high-energy part of $\Gamma(w)$ is considerably enhanced. This change makes the pion-nucleon contribution too large for the LSZ inequality to be satisfied. In other words, a pseudoresonance lying in the low-energy region removes ghosts from the theory. We agree with Frautschi[18] that a ghost probably results from inadequate approximations.

Finally we calculate $Z_N(w)$ for w in the low-energy region, keeping only the pion-nucleon intermediate states in its absorptive part. We must also assume the nonexistence of CDD poles, which present an obstacle to any function theoretical approach. Equation (2.6) then reduces to

$$Z_N(w)=1+\frac{w-m}{\pi}\int_{m+\mu}^{\infty}\frac{\tau_+^{N\pi}(w')dw'}{(w'-m)(w'-w)} -\frac{w-m}{\pi}\int_{m+\mu}^{\infty}\frac{\tau_-^{N\pi}(w')dw'}{(w'+m)(w'+w)}. \qquad (3.12)$$

The result is shown in Fig. 3. $Z_N(w)$ has a sharp maximum at almost the same energy as $\Gamma(w)$ does. Therefore, the form factor

$$K(w)=\Gamma(w)/Z_N(w) \qquad (3.13)$$

is almost constant ($0.9\sim1.1$) in the low-energy region, say, between m and $m+3\mu$. On the other hand, the nucleon propagator

$$S_F'(w)=S_F(w)/Z_N(w) \qquad (3.14)$$

is strongly damped in the pseudoresonance region.

It is not our aim to calculate the physical phase $\delta_P(w)$ by our method. Our result on $\delta_P(w)$ will be given only as a measure to check our approximation. If we denote the phase of $Z_N(w)$ on the right-hand cut by $\beta_+(w)$, we see from Eq. (3.13) that

$$\delta_+(w)=\eta_+(w)-\beta_+(w)\,, \qquad (3.15)$$

which is, in the low-energy region, equivalent to

$$\delta_P(w)=\eta_P(w)-\beta_+(w)\,. \qquad (3.16)$$

We find that the $\delta_P(w)$ thus calculated is negative, small, and almost constant ($-3.5°$ to $-5.5°$) up to $m+3\mu$, say. However, this result should not be taken too seriously, since $\delta_P(w)$ is the difference of the two big terms.

One remark will be appropriate before concluding this section. Our numerical result (3.11) is not inconsistent with the assumption, $Z_N=0$. If we determine the pion-nucleon coupling constant from the LSZ sum rule with this assumption and with the neglect of inelastic contributions, we shall obtain the value of around twenty for $g^2/4\pi$.

IV. THE PION-NUCLEON VERTEX FUNCTION WITH PION OFF THE MASS SHELL

Due to the extreme smallness of the pion mass, the pion-nucleon vertex function with pion off the mass shell is physically much more complicated than that with one nucleon off the mass shell studied in the previous sections. It seems necessary to have a survey of the former vertex function before we can start any dynamical analysis, which will not be attempted in the present work. We shall also re-examine some early approaches to this problem.

First we show, by the use of the method of Geshkenbein and Ioffe,[8] that one obtains an upper bound of about 2.0 on the coupling constant $g^2/4\pi$ if one neglects the cut of the vertex function $\Gamma(s)$ between $9\mu^2$ and

[18] S. C. Frautschi, *Regge Poles and S-matrix Theory* (W. A. Benjamin and Company, Inc., New York, 1963), Sec. 2.

$4m^2$,[2] and if $\Gamma(s)$ has no poles in the s plane with the cut extending from $4m^2$ to $+\infty$.[19] Here s denotes the invariant energy squared.

The LSZ inequality is gevin by

$$1\geq\frac{1}{\pi}\int_{4m^2}^{\infty}\tau^{N\bar{N}}(s)ds\,, \tag{4.1}$$

where

$$\tau^{N\bar{N}}(s)=\frac{2g^2}{8\pi}\frac{s(1-4m^2/s)^{1/2}}{(s-\mu^2)^2}|Y(s)|^2\,, \tag{4.2}$$

and $Y(s)$ is normalized to unity at $s=\mu^2$, by

$$Y(s)\equiv\Gamma(s)/\Gamma(\mu^2)\,. \tag{4.3}$$

We write the inequality (4.1) in the form

$$1\geq(g^2/4\pi)\Phi(Y)\,, \tag{4.4}$$

where

$$\Phi(Y)=\frac{1}{\pi}\int_{4m^2}^{\infty}\frac{s(1-4m^2/s)}{(s-\mu^2)^2}|Y(s)|^2ds\,. \tag{4.5}$$

We perform a conformal mapping

$$s\rightarrow z=\frac{(1-\beta)^{1/2}+i(x-1)^{1/2}}{(1-\beta)^{1/2}-i(x-1)^{1/2}}\,, \tag{4.6}$$

with $x=s/4m^2$ and $\beta=(\mu/2m)^2$, which transforms the s plane with the cut from $4m^2$ to $+\infty$ into the inside of the unit circle in the z plane, and the point $s=\mu^2$ to the origin $z=0$. Equation (4.5) can now be rewritten as

$$\Phi(Y)=\frac{1}{\pi}\int_0^{\pi}|Y(e^{i\theta})|^2p(\theta)d\theta\,, \tag{4.7}$$

where Y is regarded as a function of z and $p(\theta)$ is given by

$$p(\theta)=(1-\alpha)^{-1/2}u(1+u)^{-1}[1+(1-\beta)u]^{1/2} \tag{4.8}$$

with

$$u=\tan^2(\theta/2)\,. \tag{4.9}$$

$Y(z)$ has the following properties:

(I) It is analytic inside the unit circle, $|z|=1$, with the exception of possible poles on the real axis between -1 and 1 as well as the cut between z_a and 1 due to the multipion intermediate states, where z_a is the image of $s=9\mu^2$, and is given by

$$z_a=\frac{(1-\beta)^{1/2}-(1-9\beta)^{1/2}}{(1-\beta)^{1/2}+(1-9\beta)^{1/2}}(\approx 0.01) \tag{4.10}$$

(II) $Y(z^*)=\{Y(z)\}^*$ for $|z|<1$.

Hence $Y(z)$ is real on the real axis between -1 and z_a.

(III) $Y(0)=1$.

Let us introduce an auxiliary function defined by[20]

$$D(z)=\exp\left[\frac{1}{4\pi}\int_{-\pi}^{\pi}\frac{e^{i\theta}+z}{e^{i\theta}-z}\ln p(|\theta|)d\theta\right]. \tag{4.11}$$

It is regular and nonvanishing inside the unit circle. It can also be seen that $D(z^*)=\{D(z)\}^*$ for $|z|<1$ and

$$|D(re^{i\theta})|^2\rightarrow p(|\theta|)\quad\text{as}\quad r\rightarrow 1-0\,. \tag{4.12}$$

Therefore, if we write

$$Y(z)=F(z)Y_{GI}(z) \tag{4.13}$$

with

$$Y_{GI}(z)\equiv D(0)/D(z)\,, \tag{4.14}$$

Eq. (4.7) is expressed in the form

$$\Phi(Y)=\{D(0)\}^2I(F)\,, \tag{4.15}$$

where

$$I_r(F)=\frac{1}{2\pi}\int_{-\pi}^{\pi}|F(re^{i\theta})|^2d\theta\,,\quad 0\leq r\leq 1 \tag{4.16}$$

and $I(F)$ stands for $I_1(F)$. $F(z)$ has the same properties, (I), (II), and (III), as $Y(z)$ does.

We want to find the minimum of $I(F)$ over the functions $F(z)$ satisfying the following properties as well as (II) and (III):

(I′) $F(z)$ is regular inside the unit circle.

(IV) $F(z)$ belongs to the class H_2.[20,21]

(IV) means that the limit of $I_r(F)$ as $r\rightarrow 1-0$, which always exists, is *finite*:

$$\lim_{r\rightarrow 1-0}I_r(F)=1+\sum_{n=1}^{\infty}|a_n|^2<\infty\,, \tag{4.17}$$

where a_n's are the coefficients of power series expansion of $F(z)$,

$$F(z)=1+\sum_{n=1}^{\infty}a_nz^n. \tag{4.18}$$

It follows from (4.17) that

$$I(F)=\lim_{r\rightarrow 1-0}I_r(F) \tag{4.19}$$

and hence $I(F)\geq 1$. Meyman[9] showed that (IV) is satisfied if $|F(z)|<\exp(\epsilon/|1+z|)$ for any $\epsilon>0$, or equivalently, if $|\Gamma(w)|<\exp(\epsilon' w^{1/2})$ for any $\epsilon'>0$.

Evidently the minimum of $I(F)$ is obtained when $F(z)\equiv 1$. In other words, $\Phi(Y)$ takes its minimum when

[19] A similar result has been independently obtained by Meyman and Slavnov. N. N. Meyman and A. A. Slavnov, Phys. Letters **10**, 124 (1964).

[20] See, for instance, Ya. L. Geronimus, *Polynomials Orthogonal on a Circle and Interval* (Pergamon Press Inc., New York, 1960).

[21] The property (IV) is important (Ref. 9). If (IV) is replaced by the weaker property that $F(Z)$ belongs to the class L_2, which means only that $I(F)<\infty$, we can show that for any $\delta>0$ there always exists an F, $F\in L_2$ but $F\notin H_2$, for which $I(F)<\delta$, and hence that $\Phi_{\min}=0$. Note that Eq. (4.19) is not valid for such an F.

$Y(z)$ is given by $Y_{GI}(z)$. We find that

$$\Phi_{\min}=\{D(0)\}^2 =\exp\left[\frac{1}{\pi}\int_0^{\pi} \ln p(\theta)d\theta\right]. \quad (4.20)$$

It follows from (4.4) that

$$g^2/4\pi\leq\Phi^{-1}\leq\Phi_{\min}^{-1}. \quad (4.21)$$

Substitution of Eq. (4.8) into Eq. (4.20) leads to the upper bound on the coupling constant given by

$$g^2/4\pi\leq 4(1-\beta)^{1/2}[1+(1-\beta)^{1/2}]^{-1}. \quad (4.22)$$

The right-hand side is about 2.0, which is much smaller than the experimental value of 15 for $g^2/4\pi$. This contradiction is, as we believe, due to the unphysical requirement (I′), which was used in place of (I) to derive (4.22).

Federbush *et al.*[2] calculated the vertex function by the use of a ladder approximation with the force given by one-pion exchange between a nucleon-antinucleon pair. Since the $F(z)$ corresponding to their $\Gamma(s)$ has the properties (I′), (II), (III), and (IV), their calculation encounters difficulty when $g^2/4\pi>2.0$.[22] If one takes the experimental value for $g^2/4\pi$, their N-$\bar{N}$ contribution to the LSZ sum rule should be larger than 7.5. A ghost results from this strong violation of the LSZ inequality. Addition of forces due to vector meson exchange cannot improve the situation at all. A strong suppression of their vertex function necessary to avoid ghosts can be caused only by a pole (or poles) in $\Gamma(s)$ and/or the cut of $\Gamma(s)$ between $9\mu^2$ and $4m^2$.

It might be interesting to recall at this point Redmond's prescription to eliminate a ghost pole from the propagator, if it has one. Suppose we have a pion propagator, or equivalently, a $Z_\pi(s)$ given by

$$Z_\pi(s)=1+\frac{s-\mu^2}{\pi}\int_{9\mu^2}^{\infty}\frac{\tau(s')}{s'-s}ds' \quad (4.23)$$

which leads to a ghost state. $Z_\pi(s)$ must have a ghost zero at s_0, $s_0<\mu^2$, and in place of the LSZ inequality we have

$$1\leq\frac{1}{\pi}\int_{9\mu^2}^{\infty}\tau(s)ds. \quad (4.24)$$

His procedure is to introduce a new function defined by

$$Z_{\pi R}^{-1}(s)=1-\frac{s-\mu^2}{\pi}\int_{9\mu^2}^{\infty}\frac{\sigma(s')}{s'-s}ds', \quad (4.25)$$

where

$$\sigma(s)=|Z_\pi(s)|^{-2}\tau(s). \quad (4.26)$$

[22] Very recently Uehara determined the pion-nucleon coupling constant from the requirement $Z_\pi=0$, using their vertex function. The value he obtained for $g^2/4\pi$ is about one, which satisfies the unphysical inequality (4.22) as it should. M. Uehara, Progr. Theoret. Phys. (Kyoto) (to be published).

It is easy to see that $Z_{\pi R}(s)$ has neither zeros nor poles below μ^2. We can eliminate the ghost pole in the original propagator by replacing $Z_\pi(s)$ with this new function $Z_{\pi R}(s)$. To see the effect of this replacement more clearly, we rewrite Eq. (4.25) in the form

$$Z_{\pi R}(s)=1+\frac{s-\mu^2}{\pi}\int_{9\mu^2}^{\infty}\frac{\tau_R(s')}{s'-s} \quad (4.27)$$

apart from possible CDD terms, where

$$\tau_R(s)=|Z_{\pi R}(s)|^2\sigma(s) =|Z_{\pi R}(s)/Z_\pi(s)|^2\tau(s). \quad (4.28)$$

Since $Z_{\pi R}$ does not have a ghost zero by its construction, we have the LSZ inequality,

$$1\geq\frac{1}{\pi}\int_{9\mu^2}^{\infty}\tau_R(s)ds. \quad (4.29)$$

It will be noted that the original $\Gamma(s)$ has been changed to a new function given by

$$\Gamma_R(s)=[Z_{\pi R}(s)/Z_\pi(s)]\Gamma(s). \quad (4.30)$$

If Redmond's procedure is applied to the $\Gamma(s)$ of Federbush *et al.*, for instance, the new vertex function $\Gamma_R(s)$ has a pole at the original ghost energy s_0, because $Z_\pi(s_0)=0$, $Z_{\pi R}(s_0)\neq 0$, and $\Gamma(s_0)\neq 0$. We see that the LSZ inequality is made satisfied by this new pole in $\Gamma_R(s)$.

Everything sounds all right up to this point. If we look at the form factor, however, we find it remains unchanged and still retains a ghost pole. Indeed, we have

$$K_R(s)\equiv\Gamma_R(s)/Z\pi_R(s) =\Gamma(s)/Z_\pi(s)\equiv K(s). \quad (4.31)$$

After all, the present author feels that a true theory should not need a remedy for ghosts. In fact, it seems that all the ghosts we know result either from defects of a theory (the Lee model,[23] for instance) or from inadequate approximations.

Possible necessity of a subtraction for the propagator of a nonrelativistic bound state, and hence of a pole in its vertex function with the bound state off the mass shell, was first suggested by Goebel and Sakita.[24] It was shown in I that this is indeed the case for a nonrelativistic bound state under some general conditions. We should be careful, however, in extending nonrelativistic results to a relativistic case, especially when they depend much on nonrelativistic kinematics. It seems better to proceed by assuming no subtraction for the pion propagator until we encounter an insurmountable difficulty, which we hope will not exist although we cannot be certain. This assumption forbids $\Gamma(s)$ to have a pole below μ^2. Since it cannot have poles above

[23] T. D. Lee, Phys. Rev. **95**, 1329 (1954).

[24] C. J. Goebel and B. Sakita, Phys. Rev. Letters **11**, 293 (1963).

$4m^2$ by unitarity,[4] a pole (or poles) in $\Gamma(s)$ can only appear in the interval $\mu^2<s<4m^2$. We should not dismiss another interesting possibility that $\Gamma(s)$ has a pseudoresonance between $9\mu^2$ and $4m^2$.

To summarize, the LSZ inequality requires a strong suppression of the vertex function for s above $4m^2$, which we conjecture is caused possibly by a pole in $\Gamma(s)$ at s, $\mu^2<s<4m^2$, and perhaps more likely by a pseudoresonance in $\Gamma(s)$ at s, $9\mu^2<s<4m^2$.

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APPENDIX: THE GESHKENBEIN-IOFFE UPPER BOUND IN THE FERMION CASE

We shall rederive here the upper bound on the coupling constant obtained by Geshkenbein and Ioffe[8] (GI) in the fermion case, because their original derivation, which seems somewhat cumbersome in this case, can be considerably simplified. We shall also give an explicit expression for the vertex function which minimizes the integral Φ given below.

We write the LSZ inequality, (2.19), in the form

$$1\geq (g^2/4\pi)\Phi(Y)\,, \tag{A1}$$

where

$$\Phi(Y)=\frac{3}{4\pi}\left\{\int_{m+\mu}^{\infty}\frac{\rho_+(w)}{(w-m)^2}|Y(w)|^2dw +\int_{m+\mu}^{\infty}\frac{\rho_-(w)}{(w+m)^2}|Y(-w)|^2dw\right\}. \tag{A2}$$

$Y(w)$ is analytic in the complex w plane with the two cuts extending from $m+\mu$ to $+\infty$ and from $-(m+\mu)$ to $-\infty$, is real on the real axis between the two cuts, and is normalized to unity at $w=m$. We make the conformal mapping

$$w\rightarrow z=\left[\left(\frac{1+x}{1+\alpha}\right)^{1/2}-\left(\frac{1-x}{1-\alpha}\right)^{1/2}\right]\Bigg/\left[\left(\frac{1+x}{1+\alpha}\right)^{1/2}+\left(\frac{1-x}{1-\alpha}\right)^{1/2}\right], \tag{A3}$$

with $x=w/(m+\mu)$ and $\alpha=m/(m+\mu)$. It maps the w plane with the two cuts into the inside of the unit circle in the z plane, and the point $w=m$ to the origin $z=0$. If we put $z=e^{i\theta}$ on the unit circle, x is given by

$$x=[1+\alpha+(1-\alpha)u][1+\alpha-(1-\alpha)u]^{-1}\,, \tag{A4}$$

with $u=\tan^2(\theta/2)$.

Equation (A2) can now be written in the form

$$\Phi(Y)=\frac{1}{\pi}\int_0^{\pi}|Y(e^{i\theta})|^2p(\theta)d\theta\,, \tag{A5}$$

where

$$p(\theta)=12\frac{(1-\alpha)^2}{(1+\alpha)^{1/2}}\frac{u^2}{1+u}\times\frac{[\alpha(1+\alpha)+(1-\alpha)^2u]^{1/2}[1+\alpha+\alpha u]^{1/2}}{|1+\alpha-(1-\alpha)u|[1+\alpha+(1-\alpha)u]^3}. \tag{A6}$$

Since $p(\theta)$ is singular at $u=(1+\alpha)/(1-\alpha)$, which corresponds to infinite w, it is convenient to make the transformation,

$$\tilde{Y}(z)=(1+2\alpha z+z^2)^{-1/2}Y(z)\,, \tag{A7}$$

$$\tilde{p}(\theta)=2(1+u)^{-1}|1+\alpha-(1-\alpha)u|p(\theta) =\frac{3}{2}\frac{(1-\alpha)^2}{(1+\alpha)^{1/2}}\left(\frac{4u}{1+u}\right)^2\times\frac{[\alpha(1+\alpha)+(1-\alpha)^2u]^{1/2}[1+\alpha+\alpha u]^{3/2}}{[1+\alpha+(1-\alpha)u]^3}. \tag{A8}$$

Equation (A5) is rewritten as

$$\Phi=\frac{1}{\pi}\int_0^{\pi}|\tilde{Y}(e^{i\theta})|^2\tilde{p}(\theta)d\theta. \tag{A9}$$

As we did in Sec. IV we introduce the function $D(z)$,

$$D(z)=\exp\left[\frac{1}{4\pi}\int_{-\pi}^{\pi}\frac{e^{i\theta}+z}{e^{i\theta}-z}\ln\tilde{p}(|\theta|)d\theta\right], \tag{A10}$$

which is explicitly given by

$$D(z)=(3/2)^{1/2}(1-\alpha)(1+\alpha)^{-1/4}[2v/(1+v)]^2\times[\{\alpha(1+\alpha)\}^{1/2}+(1-\alpha)v]^{1/2}[(1+\alpha)^{1/2}+\alpha^{1/2}v]^{3/2}\times[(1+\alpha)^{1/2}+(1-\alpha)^{1/2}v]^{-3}, \tag{A11}$$

where $v=(1-z)/(1+z)$.

Since the remainder of the argument goes almost the same as in Sec. IV, we shall not repeat it here. The minimum of the Φ's over the $\tilde{Y}$'s having no poles inside the unit circle [and satisfying the other properties corresponding to (II), (III), and (IV) in Sec. IV] is obtained when $\tilde{Y}(z)$ is given by $\tilde{Y}_{GI}(z)$,

$$\tilde{Y}_{GI}(z)\equiv D(0)/D(z)\,. \tag{A12}$$

We also find that

$$\Phi_{\min}=\{D(0)\}^2$$

$$=\exp\left[\frac{1}{\pi}\int_0^\pi \ln\tilde{p}(\theta)d\theta\right]$$

$$=\frac{3}{16}(1-\alpha)^2[\alpha^{1/2}+(1+\alpha)^{-1/2}(1-\alpha)]\times\left[\frac{\alpha^{1/2}+(1+\alpha)^{1/2}}{1+(1-\alpha^2)^{1/2}}\right]^3. \quad (A13)$$

This result coincides with that of Geshkenbein and Ioffe, although theirs might seem much more complex at first sight.[25] It follows that

$$g^2/4\pi\leq\Phi_{\min}^{-1}=\frac{16}{3}(1-\alpha)^{-2}[\alpha^{1/2}+(1+\alpha)^{-1/2}(1-\alpha)]^{-1}\times\left[\frac{1+(1-\alpha^2)^{1/2}}{\alpha^{1/2}+(1+\alpha)^{1/2}}\right]^3. \quad (A14)$$

For the observed mass ratio we have

$$g^2/4\pi\leq 85\,,\quad\text{or}\quad f^2\leq 0.47\,. \quad (A15)$$

[25] Their equation (18) for $\Phi_{\min}$ in the article, B. V. Geshkenbein and B. L. Ioffe, Phys. Rev. Letters **11**, 55 (1963), has two unfortunate misprints. Equation (A13) should be compared with their result given in their first paper cited in Ref. 8.

In Sec. III we have assumed that the phases of $Y(w)$ become asymptotically the same in the high-energy region as those of $Y_{\mathrm{GI}}(w)$, which is given as a function of z by

$$Y_{\mathrm{GI}}(z)=(1+2\alpha z+z^2)^{1/2}\tilde{Y}_{GI}(z)=(1+2\alpha z+z^2)^{1/2}[D(0)/D(z)]. \quad (A16)$$

A plausible reason for this assumption has been given there, although it is open to criticism. It is to be noted here that $Y_{GI}(z)$ does not have correct threshold behavior at $z=\pm1$. Indeed, it becomes infinite at these points. This defect should be cured by the function

$$F(z)=Y(z)/Y_{\mathrm{GI}}(z)\,. \quad (A17)$$

However, such a healing necessarily increases the integral Φ. Therefore, if the GI bound were only slightly larger than the experimental value of $g^2/4\pi$, introduction of a pole to the vertex function would be almost imperative. Fortunately this is not the case. That is why we have assumed no poles in $\Gamma(w)$ in Sec. III. The reasonable result we have obtained seems to be in favor of the assumption.

In this connection, it is interesting to recall that in the nonrelativistic deuteron problem the GI bound (0.19) is very close to the deuteron-nucleon coupling constant (0.16) determined from low-energy scattering parameters. Indeed, it can be shown that the deuteron vertex function has a pole, which is due to a zero of the deuteron propagator.[24,4]